Effect of magnetic field on macroscopic quantum tunneling escape time in small Josephson junctions

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We study the phenomenon of macroscopic quantum tunneling (MQT) in a finite-size Josephson junction (JJ) with an externally applied magnetic field. As is well known, the problem of MQT in a pointlike JJ is reduced to the study of the under-barrier motion of a quantum particle in the washboard potential. In the case of a finite-size JJ placed in magnetic field, this problem is considerably more complex since, besides the phase, the potential itself depends on space variables. We find the general expressions both for the crossover temperature T_0 between thermally activated and MQT regimes and the escape time $\tau_{\rm esc}$. It turns out that in the proximity of particular values of magnetic field, the crossover temperature can vary nonmonotonically.

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I. INTRODUCTION

The Josephson effect[1,2] allows the investigation of fundamental aspects of quantum phenomena such as the macroscopic quantum tunneling (MQT),[3] which has been more recently observed also in high-T_c biepitaxial *YBCO* junctions.[4,5] Interesting results have been obtained also in various *Bi*-2212 structures, particularly referred to as "intrinsic" Josephson junctions (JJ).

Usually the phenomenon of MQT is considered in a "pointlike" JJ, i.e., completely neglecting the finiteness of the junction size L (some exceptions can be found in theoretical and experimental[6] papers[7,8]). This (zero order in L) approximation is based on the assumption that the junction size is much smaller than all other related parameters of the problem such as the Josephson penetration depth $\lambda_J=(\hbar c^2/8\pi e j_c \ell_{\rm eff})^{1/2}$ and the characteristic length $L_H=\ell_H^2/\ell_{\rm eff}$ (with $\ell_H=(\hbar c/eH)^{1/2}$ as the standard quantum-mechanical magnetic length).[9] Josephson critical current density j_c is determined by the Ambegaokar-Baratoff theory.[2,10] The effective length $\ell_{\rm eff}$ depends on the relation between the thickness $d_{(i)}$ $(i=L,R)$ of the superconductive electrodes and the London penetration depth $\lambda_{(i)}$ of the bulk superconductor materials, which the electrodes are made of. In the limiting cases one can find,[11]

$$\ell_{\rm eff}=\begin{cases}\lambda_{(L)}+\lambda_{(R)}+d_{\rm ox}, & \lambda_{(i)}\ll d_{(i)}\\ d_{(L)}+d_{(R)}+d_{\rm ox}, & d_{(i)}\ll\lambda_{(i)}\end{cases}. \tag{1}$$

Thermal fluctuations in JJ produce a typical "rounding" of the Josephson current branch in the *I*-*V* curves.[12,13] Since the pioneering measurements of thermal fluctuation phenomena (to obtain such an effect), the required condition between thermal energy and Josephson coupling energy ($E_J\sim T$) was usually realized not by increasing the temperature, rather by reducing the value of E_J by applying a proper magnetic field. This, even in the case of small JJ ($L<\lambda_J$), has significant implications which become of paramount importance in MQT activation.

In this context the authors[14] reported the analysis of the role of finiteness of the junction's length L obtaining the general expression for the crossover temperature T_0 between thermally activated and MQT regimes for such JJ. The escape time $\tau_{\rm esc}$ was calculated with the exponential accuracy in the first approximation in $(L/L_H)^2$ for temperatures $T>T_0$ and for small region below T_0. It was demonstrated that the account for the junction's size results in the appearance of a strong sensitivity of the MQT process on applied magnetic field, making the crossover temperature be nonmonotonic function of it. Since magnetic field is an easily adjustable parameter, it can become an important tool in the study of such a quantum coherent phenomenon without modification of other junction parameters.

In the present paper we will proceed and develop the study of MQT in a finite-size JJ placed in magnetic field (see Fig. 1). First we will report the mean-field solutions for the effective action of such a finite-size Josephson system and find the values of the effective action at the extremal trajectories. Then, the explicit form of phase trajectories close to the extremal ones and the corresponding action functional will be found. This will allow us to find the pre-exponential factor in the expression for $\tau_{\rm esc}$ of such extended system in a wide temperature region, including the crossover point T_0. It turns out that in the vicinity of magnetic-field values $H_n=\Phi_0 n/(L\ell_{\rm eff})$ ($\Phi_0=\pi\hbar c/e$ is the magnetic-flux quantum and c is the light velocity), the escape time can vary nonmonotonically—a phenomenon which becomes the fingerprint for the experimental check of the proposed theory.

In our analysis we will suppose that $L\ll\lambda_J$ but will not impose any restrictions on the relation between L and L_H. The whole analysis of the present work is referred to con-

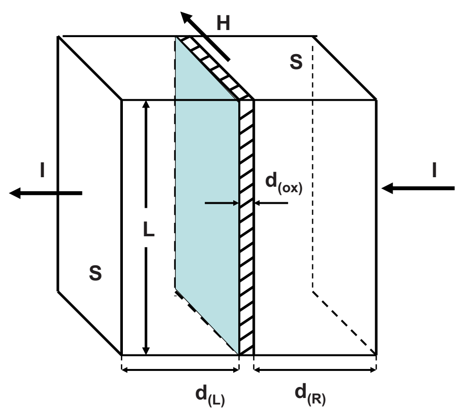

FIG. 1. (Color online) Josephson-junction geometry.

ventional (s wave) superconductors. The recent observation of MQT in high-T_c structures would also deserve great attention in connection to the role of the d-wave order-parameter symmetry but it will be considered elsewhere.

II. GENERALITIES

A. Escape time

Starting the discussion of the phenomenon of Josephson current decay in a finite-size JJ, let us recall the substance of this process in a pointlike junction.

Let us consider a current biased Josephson tunnel junction. In the framework of the capacitively shunted junction model, it can be represented by the electronic equivalent circuit, where the resistance of the external circuit (R_{ext}), the intrinsic junction capacitance (C), and the junction resistance (R_T)—assumed as a linear ohmic element—are connected in parallel.[2] The current balance in the circuit can be accounted by the following equation:

$$I = I_c \sin 2\phi + \frac{V}{R_T} + C\frac{dV}{dt}, \tag{2}$$

where

$$\phi = e\int V\,dt \tag{3}$$

is the relative phase between the two superconductors. Equation (2) can be rewritten in the form

$$M_C\frac{\partial^2\phi}{\partial t^2} + \eta\frac{\partial\phi}{\partial t} + \frac{\partial U(\phi)}{\partial\phi} = 0, \tag{4}$$

where $M_C=\hbar^2 C/e^2$, $\eta=\hbar(e^2R_T)^{-1}$, and

$$U(\phi) = -\left(\frac{\hbar I}{e}\phi + E_J\cos 2\phi\right), \quad E_J = \frac{\hbar I_c}{2e}. \tag{5}$$

Equation (4) can be considered as the equation of motion with friction (η is its viscosity) of a particle of mass M_C in the washboard potential (5) (see Fig. 2). The values of the bias current I and the critical current I_c determine the slope of the potential $U(\phi)$ and the depth of valleys.

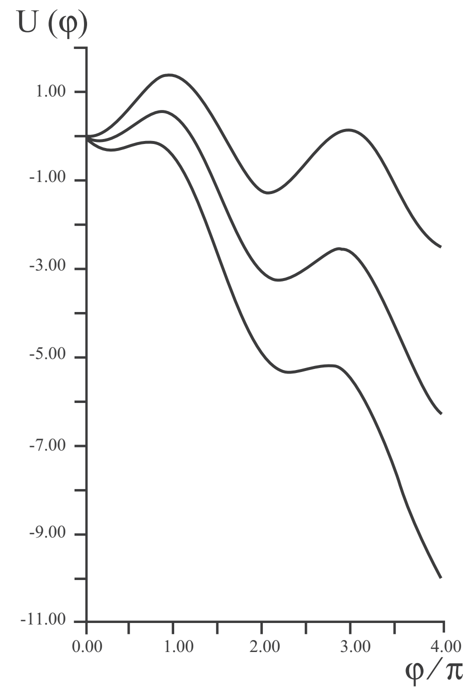

FIG. 2. Washboard potential.

Let us study the case $I<I_c$. The minima of potential (5) correspond to the metastable states of the junction. We will assume the junction viscosity to be small enough not to affect noticeably the particle oscillations in the well and its under-barrier motion. At high temperatures the thermally activated escape dominates and the result of classical Kramers problem of a particle moving in the washboard potential $U(x)$ is valid,

$$\Gamma = [\tau_{esc}^{(th)}]^{-1} \approx \frac{\omega_p}{2\pi}\exp\left[-\frac{\Delta U}{T}\right], \tag{6}$$

where the characteristic plasma frequency ω_p is given by the interpolation formula

$$\omega_p = \left[\frac{1}{M_C}\frac{d^2U(\phi)}{d\phi^2}\right]^{1/2}_{\phi_{min}} = \left(\frac{2\pi c I_c}{\Phi_0 C}\right)^{1/2}\left[1-\left(\frac{I}{I_c}\right)^2\right]^{1/4} \tag{7}$$

and $\Delta U=U_{max}-U_{min}$. The extremes of potential U_{max} and U_{min} correspond to the local maximum and local minimum of the potential energy, respectively.

When the temperature decreases, thermal activation becomes less and less probable and the process of quantum tunneling through the barrier becomes important. Near some temperature T_0 the crossover between Arrhenius law and quantum tunneling regime takes place. The latter can be considered as the under-barrier motion of the particle (instanton propagation).

At very low temperatures ($T\ll\omega_p$) the tunneling takes place only from the lowest level of the energy spectrum. When $I_c-I\ll I_c$, the escape rate can be presented in the form[15]

$$\Gamma = [\tau_{esc}^{(qm)}]^{-1} = \frac{6\omega_p}{\sqrt{\pi}}\left(\frac{6\Delta U}{\omega_p}\right)^{1/2} \exp\left[-\frac{36\Delta U}{5\omega_p}\right]. \tag{8}$$

One can see that the main difference between Eqs. (6) and (8) is in the temperature dependence of the exponent: in the classical case, Eq. (6), this is the Boltzmann activation law with the exponent equal to the barrier height divided by T, while in the case of the quantum tunneling, Eq. (8), T in the exponent has to be substituted by ω_p.

At an arbitrary temperature the combined tunneling occurs by the following scheme. First, the particle excites in the thermal activation manner and at some moment gets up to the energy level E_n (calculated neglecting tunneling) and then, by means of quantum tunneling, passes through the barrier. The total tunneling probability is determined by the sum over all quantum levels of corresponding products of the classical and quantum probabilities,

$$\Gamma \sim \sum_n \exp\left(-\frac{E_n}{T}\right)\exp[-A(E_n)]. \tag{9}$$

Here

$$A(E_n) = \oint \sqrt{2M_C[E_n - U(\phi)]d\phi} \tag{10}$$

is the action corresponding to the under-barrier motion of the particle with energy E_n.

One can say that quantum tunneling occurs when the particle reaches such a level E_{tun} at which the probability of the direct quantum tunneling through the barrier becomes larger than the probability of the activation jump on the higher energy levels with further tunneling through the barrier. Quantitatively this can be formulated as the condition for the extremum of the exponent in Eq. (9),

$$\left.\frac{\partial A(E)}{\partial E}\right|_{E_{tun}} = -\frac{1}{T}. \tag{11}$$

Condition (11) implies that the period of the particle oscillation in the inverted potential is equal to $1/T$. With the growth of temperature, the energy E_{tun} increases and when $T = \omega_p/2\pi$ it reaches the barrier height. At higher temperatures the classical activation escape scheme is realized.

The value of the escape time τ_{esc} of the finite-size JJ in magnetic field can be determined in the framework of the general method of functional integration. In this approach the escape rate of the MQT, which is defined by the imaginary part of the free energy $\tau_{esc}^{-1} = 2\,\mathrm{Im}\,F$, can be expressed in terms of the partition function of the system[16]

$$F = -T \ln Z, \tag{12}$$

where the latter is defined by the functional integral over all possible "surfaces" $\varphi(\mathbf{r},\tau)$,

$$Z = \int D\varphi(\mathbf{r},\tau)\exp\{-A[\varphi(\mathbf{r},\tau)]\}, \tag{13}$$

where $\varphi(\mathbf{r},\tau)$ is the phase difference on the junction taken at the point r and at the imaginary time moment τ. Here $A[\varphi(\mathbf{r},\tau)]$ is the effective action of such extended system.

The problem of MQT makes sense only within the framework of the quasiclassical approximation, i.e., in assumption that the value of escape time considerably exceeds all characteristic time scales of the internal motions. In this case, the imaginary part of the partition function is small in comparison with the real one, hence,

$$\tau_{esc}^{-1} = 2T\frac{\mathrm{Im}\int D\varphi(\mathbf{r},\tau)\exp\{-A[\varphi(\mathbf{r},\tau)]\}}{\mathrm{Re}\int D\varphi(\mathbf{r},\tau)\exp\{-A[\varphi(\mathbf{r},\tau)]\}}. \tag{14}$$

The presence of the nonzero imaginary part of partition function in the numerator of Eq. (14) is related to the fact that one of the action's $A[\varphi(\mathbf{r},\tau)]$ extremal trajectories $[\varphi^{sdl}(\mathbf{r},\tau)]$ is of the saddle type. As a matter of the fact, the negative eigenvalue corresponds to one of the modes around this saddle trajectory [see below Eqs. (20) and (21)]. As a consequence, in the process of the functional integration in Eq. (14), the contour of integration over this mode should be shifted to the imaginary axis,[16–18] which leads to the appearance of Im Z. What concerns the real part of the partition function is that it has be calculated at the minimal trajectory $\varphi^{min}(\mathbf{r},\tau)$.

For high enough temperatures $T > T_0$, or in narrow vicinity of T_0 ($|T-T_0| \ll T_0$), both trajectories $\varphi^{sdl}(\mathbf{r})$ and $\varphi^{min}(\mathbf{r})$ turn out to be time independent[16,19] and the exponential factor in τ_{esc}^{-1} [in analogy with Eq. (6)] is determined by the expression

$$\tau_{esc}^{-1} \sim \exp(-\Delta A_{min}^{sdl}),$$

$$\Delta A_{min}^{sdl} = A[\varphi^{sdl}(\mathbf{r})] - A[\varphi^{min}(\mathbf{r})]. \tag{15}$$

In the assumption of zero viscosity, one can obtain both thermally activation and MQT escaped times (the latter even with pre-exponential factor accuracy).[16] For temperatures above but not too close to T_0, it is read as

$$\tau_{esc}^{-1} = T_0\frac{\sinh\left(\frac{\omega_p}{2T}\right)}{\sin(\pi T_0/T)}\exp(-\Delta A_{min}^{sdl}). \tag{16}$$

Hence, in order to find the escape time [Eqs. (15) and (16)], one has to calculate the values of action at trajectories φ^{sdl} and φ^{min}. The crossover temperature T_0 turns to be the bifurcation point, below which the time-dependent solution for the saddle-point trajectory $\varphi^{sdl}(\mathbf{r},\tau)$ becomes more favorable than the static one and the definition of escape time presents more sophisticated problems.

B. Effective action

The effective action of a JJ placed in external magnetic field H as a function of flowing through the junction current I in the first order in transparency can be written down in the most general form basing on the results of[20,21]

$$A[\varphi(\mathbf{r},\tau)]=\frac{1}{S}\int_{-1/2T}^{1/2T}d\tau\int d^2\mathbf{r}\left\{\frac{C}{2e^2}\left[\frac{\partial\varphi(\mathbf{r},\tau)}{\partial\tau}\right]^2-\frac{\hbar}{e}I\varphi(\mathbf{r},\tau)\right.$$
$$-\frac{\pi\hbar}{2R_Ne^2}\int_{-1/2T}^{1/2T}d\tau_1\{\{1-\cos[\varphi(\mathbf{r},\tau)-\varphi(\mathbf{r},\tau_1)]\}$$

$$\alpha_L(\tau-\tau_1)\alpha_R(\tau-\tau_1)+\cos[\varphi(\mathbf{r},\tau)+\varphi(\mathbf{r},\tau_1)]$$
$$\times\beta_L(\tau-\tau_1)\beta_R(\tau-\tau_1)\}+\frac{\pi\hbar T^2}{R_{\mathrm{sh}}\,e^2}$$

$$\left.\int_{-1/2T}^{1/2T}d\tau 1\frac{\sin^2\{[\varphi(\mathbf{r},\tau)-\varphi(\mathbf{r},\tau_1)]/2\}}{\sin^2[\pi T(\tau-\tau_1)]}\right\}$$
$$+\frac{\hbar^2c^2}{8\pi e^2\ell_{\mathrm{eff}}}\int_{-1/2T}^{1/2T}d\tau\int d^2\mathbf{r}\left[\frac{\partial\varphi(\mathbf{r},\tau)}{\partial\mathbf{r}}-\frac{e\ell_{\mathrm{eff}}}{\hbar c}(\mathbf{H}\times\mathbf{n})\right]^2. \tag{17}$$

Here C is the junction capacitance, R_{N} is the tunnel resistance, R_{sh} is the shunt resistance, and $\alpha_{(L,R)}(\tau)$ and $\beta_{(L,R)}(\tau)$ are integrated over the energy variable normal and anomalous Green's functions, respectively. The integrals are carried out over the imaginary time and the junction area. The first term corresponds to the kinetic energy of the junction. The second and fourth terms describe the contribution of the potential energy to the action. The third term corresponds to the capacitance renormalization,[16] while the last term in Eq. (17) accounts for the finite size of the junction and the magnetic-field contribution to the effective action.[22]

The natural assumption $T_0 \ll T_c$, side by side with the assumed above condition $L \ll \lambda_J$, allows one to considerably simplify the general expression Eq. (17), which was done in the leading approximation in $(L/\lambda_J)^2$ in Ref. 14. It was shown that in these conditions, the third term in Eq. (17) disappears being reduced to renormalization of the capacity C in the first term

$$C^*=C+\frac{\pi\hbar}{2R_{\mathrm{N}}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\frac{\partial\alpha_L}{\partial\omega}\right)\left(\frac{\partial\alpha_R}{\partial\omega}\right) \tag{18}$$

and changing in the definite way the shape of $\varphi(\mathbf{r},\tau)$.

Variation of Eq. (17) on φ results in getting the quasiclassical equations of motion,[11] which define the extremal trajectories,

$$\left.\frac{\delta A[\varphi]}{\delta\varphi}\right|_{\varphi=\varphi^{\mathrm{extr}}(\mathbf{r},\tau)}=0. \tag{19}$$

Near the extremal trajectory $\varphi^{\mathrm{extr}}(\mathbf{r},\tau)$, the deviation of the function $\varphi(\mathbf{r},\tau)$ can be represented in the form of expansion by normalized functions $\varphi_n^k(\mathbf{r},\tau)$,

$$\varphi(\mathbf{r},\tau)=\varphi^{\mathrm{extr}}(\mathbf{r},\tau)+\sum B_n^k\varphi_n^k(\mathbf{r},\tau), \tag{20}$$

which are the eigenfunctions of the equation

$$\frac{\delta^2A[\varphi]}{\delta\varphi^2}\varphi_n^k(\mathbf{r},\tau)=\Lambda_n^k\varphi_n^k(\mathbf{r},\tau). \tag{21}$$

In this representation the action (17) near the extremal trajectory $\varphi^{\mathrm{extr}}(\mathbf{r},\tau)$ is presented by Gaussian type functional integral over functions $\varphi_n^k(\mathbf{r},\tau)$, which can be carried out analytically. Thus, the problem of definition of the value of escape time is reduced to finding of the eigenvalues Λ_n^k of the Eq. (21).[16]

We will restrict our consideration within the two regions of temperatures: (i) $T>T_0$ and (ii) $|T-T_0|\ll T_0$. This choice is related to the fact that in both these regions, one can use the functions $\varphi(\mathbf{r},\tau)$ in the form (20) with time independent $\varphi^{\mathrm{extr}}(\mathbf{r})$. The width of the crossover region between thermal activation and MQT regimes turns to be much smaller than T_0 and will be estimated below.

III. EXTREMAL TRAJECTORIES IN STATIC APPROXIMATION

Let us find the explicit form of the extremal trajectory φ^{extr} in static approximation, i.e., in the case when it can be considered as time independent. Moreover, in the geometry under consideration, when the magnetic field is applied along the junction, the phase depends only on one coordinate x. Substitution of $\varphi^{\mathrm{extr}}(x,\tau)\equiv\varphi^{\mathrm{extr}}(x)$ in Eq. (19) leads to the equation

$$-\lambda_J^2\frac{\partial^2\varphi^{\mathrm{extr}}(x)}{\partial x^2}+\frac{1}{2}\sin 2\varphi^{\mathrm{extr}}(x)=\frac{I}{2j_cS}. \tag{22}$$

We will look for its solutions in the form

$$\varphi^{\mathrm{extr}}(x)=\varphi_0-\frac{x}{L_H}+\tilde{\varphi}(x), \tag{23}$$

where φ_0 is a constant and $\langle\tilde{\varphi}(x)\rangle=0$. Corresponding boundary conditions are

$$\left.\frac{\partial\tilde{\varphi}}{\partial x}\right|_{x=-L/2}=\left.\frac{\partial\tilde{\varphi}}{\partial x}\right|_{x=L/2}=0. \tag{24}$$

One can look for solutions of Eq. (22) in the frameworks of the perturbation theory by the parameter $(L/\lambda_J)^2$. In the first approximation one can rewrite Eq. (22) in the form

$$-\lambda_J^2\frac{\partial^2\tilde{\varphi}(x)}{\partial x^2}=\frac{I}{2Sj_c}-\frac{1}{2}\sin\left[2\left(\varphi_0-\frac{x}{L_H}\right)\right]. \tag{25}$$

This equation with the given above boundary conditions is easily solvable,

$$\tilde{\varphi}(x)=\frac{1}{2}\left(\frac{L_H}{2\lambda_J}\right)^2\left\{\left[\sin\left(\frac{2x}{L_H}\right)-\frac{2x}{L_H}\cos\frac{L}{L_H}\right]\cos 2\varphi_0\right.$$
$$+\left[\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)-\cos\left(\frac{2x}{L_H}\right)\right.$$
$$\left.\left.+\sin\left(\frac{L}{L_H}\right)\left(\frac{L}{6L_H}-\frac{2x^2}{LL_H}\right)\sin 2\varphi_0\right]\right\}. \tag{26}$$

The phase $\varphi(x,\varphi_0)$ is a periodic function of φ_0 with the period π. It turns out that some critical value of the current

$I_{cr}(H)$ exists such that for all $I<I_{cr}$, one can find for each period two solutions for φ_0. The exception presents narrow regions of certain magnetic-field values, which will be found and investigated below. It will be shown that for such regions, four different solutions for φ_0 exist in the interval $(-\pi/2, \pi/2)$. Two solutions correspond to the minimum of action while the other two to its saddle point. When $I=I_{cr}$ pair solutions (one corresponding to the minimum, the other to the saddle point of the action) confluent in one, while for $I>I_{cr}$ the static solution does not exist at all. In the simple case of a pointlike junction, one can find

$$\varphi_{00}^{\min} = \frac{1}{2}\arcsin\left(\frac{I}{j_c S}\right) \tag{27}$$

and

$$\varphi_{00}^{\text{sdl}} = \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{I}{j_c S}\right). \tag{28}$$

When the junction has finite width, the analysis is more complicated. Nevertheless the smallness $L \ll \lambda_J$ allows us to find the relation between φ_0 and I, i.e., to find $\varphi^{\text{extr}}(x,I)$. Indeed, integrating Eq. (22) over x, one can obtain

$$I = j_c S\int_{-L/2}^{L/2}\frac{dx}{L}\sin\left[2\varphi_0 - \frac{2x}{L_H} + 2\tilde{\varphi}(x)\right]. \tag{29}$$

Substituting in this expression $\tilde{\varphi}(x)$ according to Eq. (26) and performing integration, one finds the equation which implicitly relates φ_0 to I,

$$\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\sin 2\varphi_0 + \frac{1}{2}\left(\frac{L_H}{2\lambda_J}\right)^2, \quad \kappa\left(\frac{L_H}{L}\right)\sin 4\varphi_0 = \frac{I}{j_c S}, \tag{30}$$

where

$$\kappa\left(\frac{L_H}{L}\right) = 2\left(\frac{L_H}{L}\right)^2\sin^2\left(\frac{L}{L_H}\right) + \cos^2\left(\frac{L}{L_H}\right) - \frac{3L_H}{2L}\sin\left(\frac{2L}{L_H}\right) - \frac{1}{3}\sin^2\left(\frac{L}{L_H}\right). \tag{31}$$

The critical current $I_{cr}(H)$ is determined by the value $I(\hat{\varphi}_0)$, where the point $\hat{\varphi}_0$ is the solution of the equation

$$\left.\frac{\partial I(\varphi_0)}{\partial \varphi_0}\right|_{\varphi_0=\hat{\varphi}_0} = 0, \quad I_{cr}(H) = I(\hat{\varphi}_0). \tag{32}$$

IV. VALUE OF EFFECTIVE ACTION AT THE EXTREMAL TRAJECTORIES

Let us find the value of effective action of the finite-size JJ on the extremal trajectory $\varphi^{\text{extr}}(x)$. The knowledge of $A[\varphi^{\text{extr}}(x)]$ will allow us to find the value of escape time with the exponential accuracy in the wide interval of temperatures above T_0 or close enough to this bifurcation point ($|T-T_0| \ll T_0$).[16] Further definition of the pre-exponential factor will require one to perform the functional integration in Eq. (14) over the trajectories close to the extremal one.

In the case when the phase trajectory $\varphi(x,\tau) \equiv \varphi(x)$ does not depend on imaginary time (static approximation), Eq. (17) is considerably simplified,

$$A[\varphi(x)] = \frac{\hbar S j_c}{eLT}\int_{-L/2}^{L/2} dx\left\{-\frac{I}{Sj_c}\varphi(x) - \frac{1}{2}\cos 2\varphi(x) + \lambda_J^2\left[\frac{\partial\varphi(x)}{\partial x} + \frac{1}{L_H}\right]^2\right\}. \tag{33}$$

The straightforward integration of this expression with the phase $\varphi(x)$ determined from the Eqs. (23) and (26) results in

$$A(\varphi_0) = -\frac{\hbar I}{eT}\varphi_0 - \frac{\hbar S j_c}{2eT}\left(\frac{L_H}{L}\right)\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \frac{\hbar S j_c}{8eT}\left(\frac{L_H}{2\lambda_J}\right)^2\left[\mu\left(\frac{L_H}{L}\right) - \kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0\right], \tag{34}$$

with

$$\mu\left(\frac{L_H}{L}\right) = 2\left\{\left[\left(\frac{L_H}{L}\right)^2 + \frac{1}{3}\right]\sin^2\left(\frac{L}{L_H}\right) - 1\right\}. \tag{35}$$

This expression already gives, in explicit form, the value of action both for minimal and saddle trajectories [determined as a function of current and magnetic field by Eq. (30)]. Let us mention that Eq. (30) could be obtained from Eq. (34) by deriving the action and equating this derivative to zero: $\partial A(\varphi_0)/\partial\varphi_0 = 0$. Moreover, by calculating the second derivative of the action (34), one can find another useful relation,

$$\frac{\partial^2 A(\varphi_0)}{\partial^2\varphi_0} = \frac{2\hbar S j_c}{eT}\left[\left(\frac{L_H}{L}\right)\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0\right]. \tag{36}$$

Note that both Eqs. (30) and (34) are valid for any value of magnetic field, even in the region where the second (correction) term in Eq. (30) becomes of the order or even larger than the first one. Equation (32) for the value of $\hat{\varphi}_0$ can be written in the form

$$\delta_1 \cos 2\hat{\varphi}_0 + 2\delta_2\cos 4\hat{\varphi}_0 = 0, \tag{37}$$

where

$$\delta_1 = \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right) \quad \text{and} \quad \delta_2 = \frac{1}{2}\left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right). \tag{38}$$

Solutions of Eq. (37) are

$$\cos 2\hat{\varphi}_0 = -\frac{\delta_1}{4\delta_2} \pm \sqrt{\left(\frac{\delta_1}{4\delta_2}\right)^2 + \frac{1}{2}}. \tag{39}$$

In the case when $|\delta_1/\delta_2| > 1$, the only physically sensible solutions are given by the equation

$$\cos 2\hat{\varphi}_0 = \frac{\delta_1}{4\delta_2}\left[\sqrt{1+8\left(\frac{\delta_2}{\delta_1}\right)^2}-1\right]. \quad (40)$$

When $|\delta_1/\delta_2|<1$, the solutions corresponding to both signs in Eq. (39) can be realized. This is the hysteresis domain; the type of solution here depends on the prehistory of magnetic-field variation. At special points $H_n = \pi\hbar cn/(eL\ell_{\text{eff}})$, where $L=\pi n L_H$ $(n=0,1,2\ldots)$, both states on each period start to be equivalent.

Let us return to Eq. (30). It can be rewritten by means of the functions $\delta_1(H)$ and $\delta_2(H)$ [see Eq. (38)] as

$$\delta_1 \sin 2\varphi_0 + \delta_2 \sin 4\varphi_0 = \frac{I}{j_c S}. \quad (41)$$

This equation can be solved exactly in the algebraic functions but for simplicity we will find its solutions $\varphi_0^{\min}$ and φ_0^{sdl} in the framework of the perturbation theory under the assumption $|\delta_1| \gg |\delta_2|$. Simple algebra leads to the result

$$\frac{1}{2}\arcsin\left(\frac{I}{j_c S\delta_1}\right) - \frac{\delta_2}{\delta_1}\left(\frac{I}{j_c S\delta_1}\right) = \begin{cases}\varphi_0^{\min}, & \text{if } I/j_c S\delta_1 > 0\\ \varphi_0^{\text{sdl}}, & \text{if } I/j_c S\delta_1 < 0\end{cases} \quad (42)$$

and

$$\frac{\pi}{2}\text{sign}\left(\frac{I}{j_c S\delta_1}\right) - \frac{1}{2}\arcsin\left(\frac{I}{j_c S\delta_1}\right) - \frac{\delta_2}{\delta_1}\left(\frac{I}{j_c S\delta_1}\right) = \begin{cases}\varphi_0^{\text{sdl}}, & \text{if } I/j_c S\delta_1 > 0\\ \varphi_0^{\min}, & \text{if } I/j_c S\delta_1 < 0\end{cases}. \quad (43)$$

The Eqs. (42) and (43) were obtained with the help of Eq. (36). Inserting the values of $\varphi_0^{\min}$ and φ_0^{sdl} to Eq. (34), one can obtain the final expression for the difference of actions on the extremal trajectories,

$$\Delta A_{\min}^{\text{sdl}} = A(\varphi_0^{\text{sdl}}) - A(\varphi_0^{\min}) = -\frac{\hbar I}{eT}\left(\pi/2 - \arcsin\left|\frac{I}{j_c S\delta_1}\right|\right) + \frac{\hbar j_c S}{eT}\sqrt{1-\left(\frac{I}{j_c S\delta_1}\right)^2}\left(\frac{L_H}{L}\right)\sin\left(\frac{L}{L_H}\right) \times\left[1+2\left(\frac{\delta_2}{\delta_1}\frac{I}{j_c S\delta_1}\right)^2\right]\text{sign}\left(\frac{I}{j_c S\delta_1}\right), \quad (44)$$

which determines in explicit form the exponential factor of the escape time (15). Looking at it one can notice the nontrivial oscillatory type dependence of the escape time on the value of external magnetic field, which we will discuss in Sec. VI of this paper.

V. VALUE OF EFFECTIVE ACTION AT TRAJECTORIES CLOSE TO THE EXTREMAL (PRE-EXPONENTIAL FACTOR)

The expression (34) obtained above allows one to determine the escape rate with the exponential accuracy, which indeed was done above. Determination of the pre-exponential factor is a more delicate task, which requires knowledge of the shape of trajectories close to the extremal one with further functional integration of the action over them. Now we pass to perform this program.

In real situation the bifurcation point T_0 lies always much below the critical temperature $T_0 \ll T_c$. For temperatures $T_0 < T \ll T_c$ or $|T_0 - T| \ll T_c$, the general expression (17) can be considerably simplified,

$$A[\varphi(x,\tau)] = \frac{1}{L}\int_{-1/2T}^{1/2T} d\tau \int_{-L/2}^{L/2} dx \left\{\frac{C}{2e^2}\left[\frac{\partial\varphi(x,\tau)}{\partial\tau}\right]^2 - \frac{\hbar}{e}I\varphi(x,\tau) - \frac{\hbar j_c S}{2e}\cos[2\varphi(x,\tau)] + \frac{\hbar}{4\pi R_{\text{sh}} e^2}\int_{-1/2T}^{1/2T} d\tau_1\left[\frac{\varphi(x,\tau)-\varphi(x,\tau)}{\tau-\tau_1}\right]^2 + \frac{\hbar j_c S}{2e}\lambda_J^2\left[\frac{\partial\varphi(x,\tau)}{\partial x} - \frac{e\ell_{\text{eff}}H_{\text{ext}}^2}{\hbar c}\right]^2\right\}. \quad (45)$$

Let us find the solutions of Eq. (21) in the vicinity of both time independent extremal trajectories (surfaces). We will look for them in the form

$$\varphi_n^k(x,\tau) = \sqrt{T}\exp(i2\pi T n\tau)\chi_n^k(x). \quad (46)$$

Substitution of this expression in Eqs. (17)–(21) leads to equations for $\chi_n^k(x)$,

$$-\lambda_J^2\frac{\partial^2\chi_n^k(x)}{\partial x^2} + \{\zeta(n^2 + Q_{\text{sh}}|n|) + \cos[2\varphi^{\text{extr}}(x)]\}\chi_n^k(x) = \frac{e}{2\hbar j_c}\Lambda_n^k\chi_n^k(x), \quad (47)$$

where we have introduced the Q-factor $Q_{\text{sh}}^{-1} = 2\pi T R_{\text{sh}} C^*/\hbar$ and the parameter $\zeta = 2\pi^2 T^2 C^*/(e\hbar j_c S)$.

Note that at the point T_0, one has

$$\Lambda_{\pm 1}^0(T_0) = 0, \quad (48)$$

since at this point the first time-dependent solution for the extremal trajectory appears.

Let us recall that Eq. (47) is valid for both extremal trajectories φ^{sdl} and $\varphi^{\min}$. We will look for its solutions in the form of perturbation-theory series in parameter $(L/\lambda_J)^2$. There are two sets of corresponding eigenfunctions. In the zero-order approximation, they can be odd or even in x.

For even values of k $(k=2N, N=0,1,2\ldots)$, we have

$$\chi_{\pm n}^{2N}(x) = \cos\left(\frac{2\pi N}{L}x\right) + \gamma_{\pm n}^{2N}(x),$$

$$\int_{-L/2}^{L/2} dx \cos\left(\frac{2\pi N}{L}x\right)\gamma_{\pm n}^{2N}(x) = 0. \quad (49)$$

For odd values of k $(k=2N+1, N=0,1,2\ldots)$, the eigenfunctions have the form

$$\chi_{\pm n}^{2N+1}(x) = \sin\left[\frac{\pi(2N+1)}{L}x\right] + \gamma_{\pm n}^{2N+1}(x),$$

$$\int_{-L/2}^{L/2} dx \sin\left[\frac{\pi(2N+1)}{L}x\right]\gamma_{\pm n}^{2N+1}(x) = 0. \tag{50}$$

The last integrals in Eqs. (49) and (50) express the relations of orthogonality between the first-order correction $\gamma_{\pm n}^{k}(x)$ and corresponding zero approximation solution. The functions γ are supposed to be small enough: $|\gamma_{\pm n}^{k}(x)| \ll 1$.

For $k \neq 0$, one can restrict consideration by the main approximation only and get from Eqs. (48)–(50) the following expression for the eigenvalues $\Lambda_{\pm n}^{k}$:

$$\frac{e}{2\hbar j_c}\Lambda_{\pm n}^{k} = \lambda_J^2\left(\frac{\pi k}{L}\right)^2 + \zeta(n^2 + Q_{\text{sh}}|n|) + \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\left[1 + \frac{1}{1-\left(\frac{L_H}{L}\pi k\right)^2}\right]\cos(2\varphi_0). \tag{51}$$

For the eigenvalues with $k=0$ ($\Lambda_{\pm n}^{0}$), we have to find the eigenvalues up to the first correction term in the parameter $(L/\lambda_J)^2$. From Eqs. (48) and (49) follows the equation for $\gamma_{\pm n}^{0}(x)$,

$$-\lambda_J^2\frac{\partial^2\gamma_{\pm n}^{0}(x)}{\partial x^2} + \cos\left(2\varphi_0 - \frac{2x}{L_H}\right) - \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 = 0. \tag{52}$$

Its solution, in view of the boundary-condition Eq. (24), is

$$\gamma_{\pm n}^{0}(x) = \left(\frac{L_H}{2\lambda_J}\right)^2\left[\frac{2x}{L_H}\cos\left(\frac{L}{L_H}\right) - \sin\left(\frac{2x}{L_H}\right)\right]\sin 2\varphi_0 - \left(\frac{L_H}{2\lambda_J}\right)^2\left[\cos\left(\frac{2x}{L_H}\right) - \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right) + \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\left(\frac{2x^2}{L_H^2} - \frac{L^2}{6L_H^2}\right)\right]\cos 2\varphi_0. \tag{53}$$

From Eqs. (47), (49), and (53) one finds the value of $\Lambda_{\pm n}^{0}$ with the first correction terms [compare to Eq. (51)],

$$\frac{e}{2\hbar j_c}\Lambda_{\pm n}^{0} = \zeta(n^2 + Q_{\text{sh}}|n|) + \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0. \tag{54}$$

One can notice that the eigenvalue Λ_0^0 in the vicinity of the saddle-point trajectory is negative. This property is an obvious consequence of the Eqs. (36) and (37) and this fact results in the appearance of the imaginary part of the partition function (13).

Substitution of $n=1$ to Eq. (54) gives us the explicit definition of the crossover temperature T_0,[14]

$$\zeta(T_0)[1 + Q_{\text{sh}}(T_0)] + \frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0 = 0, \tag{55}$$

where $\varphi_0 = \varphi^{\text{sdl}}$ and it is the function of external current I and magnetic field H [see Eqs. (30)–(42)]. For the critical value of current, where $\partial I/\partial\varphi_0 = 0$, $T_0 = 0$. This equality follows immediately from Eq. (55). Note that our parameter of perturbation theory is $(L_H/\lambda_J)^2$ and the corrections to eigenfunctions [Eqs. (49) and (53)] are small by this parameter for all values of the external current I and magnetic field H. That means that the Eq. (30) is valid even in the vicinity of the points where $\sin(L/L_H)=0$. It is important that in these regions both terms in the right-hand side of Eq. (30) are of the same order and the nontrivial dependence of $\varphi_0(I,H)$ arises since Eq. (30) is equivalent to the fourth order equation and in the considered region, all its coefficients turn out to be of the same order.

Now one can write down the expression for the effective action (45), valid near both extremal trajectories. It is enough to substitute in Eq. (45) the function $\varphi(x,\tau)$ in the form of Eq. (20) with $\varphi_n^k(x,\tau)$, defined by Eqs. (46), (49), and (53). Since the eigenvalues $\Lambda_{\pm 1}^0$ tend to zero as $T \to T_0$ for saddle-point trajectory, we have to keep in the expansion of the effective action over the coefficients $\{B_{\pm 1}^0\}$ all terms up to the fourth order, keeping also the products of the type $[(B_1^0)^2 B_{-2}^k, (B_{-1}^0)^2 B_2^k]$. In result of integrating over x and τ (see Appendix A), the action (45) takes the explicit form as the function of coefficients B_n^k,

$$A[B_0^0, B_1^0, B_{-1}^0, ..B_n^k..] = A[\varphi_{\text{extr}}(x)] - \gamma_1[2B_1^0 B_{-1}^0 B_0^0 + (B_1^0)^2 B_{-2}^0 + (B_{-1}^0)^2 B_2^0] - \gamma_2(B_1^0)^2(B_{-1}^0)^2 + \frac{1}{2}\sum_{n,k}\Lambda_n^k|B_n^k|^2. \tag{56}$$

Calculation of the coefficients $\gamma_{1,2}$ is cumbersome but straightforward. It is necessary to remember that the functions $\chi_{\pm n}^k(x)$, before being used in Eq. (45), should be normalized. In result one finds

$$\gamma_1 = \frac{2\hbar j_c}{e}\left(\frac{T}{S}\right)^{1/2}\left[\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\sin 2\varphi_0 + 2\left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\sin 4\varphi_0\right], \tag{57}$$

$$\gamma_2 = \frac{2\hbar j_c}{e}\left(\frac{T}{S}\right)\left\{\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \frac{1}{2}\left(\frac{L_H}{2\lambda_J}\right)^2\left[5\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0 + 3\mu\left(\frac{L_H}{L}\right)\right]\right\}. \tag{58}$$

VI. OSCILLATIONS OF THE ESCAPE TIME VS MAGNETIC FIELD

Now we are ready to calculate the escape time of the "small" Josephson junctions (JJ), which is given by Eq. (14). The imaginary part of the partition function Im Z in Eq. (14) is determined by the integral over trajectories close to the saddle-point trajectory $\varphi^{\text{sdl}}(x)$ [see Eq. (20)]. Corresponding expression for the action was already obtained above and it is given by Eq. (56). The functional integral in Eq. (14) is reduced now to the integration over all coefficients B_n^k.

Let us start from the integration over the coefficient B_0^0. Since the eigenvalue Λ_0^0 in the vicinity of the saddle-point trajectory is negative, it requires special considerations. In order to get the finite answer, one has (before integration) to perform analytical continuation $B_0^0 \to ib$ and (only after that) carry out the integral

$$\int_{-\infty}^{\infty}\frac{dB_0^0}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\Lambda_0^0(B_0^0)^2+2\gamma_1 B_0^0(B_1^0B_{-1}^0)\right]$$
$$\to i\int_{-\infty}^{\infty}\frac{db}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}|\Lambda_0^0|b^2+2i\gamma_1 b(B_1^0B_{-1}^0)\right]$$
$$=\frac{i}{\sqrt{|\Lambda_0^0|}}\exp\left[-\frac{2\gamma_1^2}{|\Lambda_0^0|}(B_1^0B_{-1}^0)^2\right]. \tag{59}$$

Integration of the coefficients $B_{\pm1}^0$ will yet leave it for further consideration and now perform that one over the coefficients $B_{\pm2}^0$, in accordance with the formula

$$\int_{-\infty}^{\infty}\frac{d^2B_{\pm2}^0}{2\pi}\exp\{-\Lambda_2^0(B_2^0)^2+\gamma_1[(B_1^0)^2B_{-2}^0+(B_{-1}^0)^2B_2^0\}$$
$$=\frac{1}{2\Lambda_2^0}\exp\left[\frac{\gamma_1^2}{\Lambda_2^0}(B_1^0B_{-1}^0)^2\right]. \tag{60}$$

Integrations over the remaining coefficients besides are of the canonical Gaussian type and can be easily performed.

What concerns the calculation of the real part of the partition function Re Z is determined by integrating over the trajectories passing close to the minimal trajectory $\varphi^{\min}(x)$ and in this case, one can take the action (56) only with quadratic accuracy over B_n^k.

Performing the above discussed integrations in real and imaginary parts of the partition function (see Ref. 16), one finds the escape time for high enough temperatures $T>T_0$ or in the narrow vicinity of T_0 ($|T-T_0|\ll T_0$),

$$\tau_{\rm esc}^{-1}=2T_0\exp(-\Delta A_{\min}^{\rm sdl})\cdot$$
$$\left\{\frac{1}{2\sqrt{|\Lambda_0^0|}}\int_{-\infty}^{\infty}\frac{d^2B_1^0}{2\pi}\exp\left\{-\Lambda_1^0|B_1^0|^2-|B_1^0|^4\left[-\gamma_2+\gamma_1^2\left(\frac{2}{|\Lambda_0^0|}-\frac{1}{\Lambda_2^0}\right)\right]\right\}\right\}_{\rm sdl}\{\sqrt{|\Lambda_0^0|}\}_{\min}Y_1Y_2, \tag{61}$$

with

$$Y_1=\left\{\prod_{k=1}^{\infty}\frac{1}{\sqrt{\Lambda_0^k}}\left[\prod_{n=1}^{\infty}\frac{1}{(2\Lambda_n^k)}\right]\right\}_{\rm sdl}\Big/\left[\prod_{k=1}^{\infty}\frac{1}{\sqrt{\Lambda_0^k}}\prod_{n=1}^{\infty}\frac{1}{(2\Lambda_n^k)}\right]_{\min}, \tag{62}$$

$$Y_2=\left[\prod_{n=2}^{\infty}\frac{1}{(2\Lambda_n^0)}\right]_{\rm sdl}\Big/\left[\prod_{n=1}^{\infty}\frac{1}{(2\Lambda_n^0)}\right]_{\min}. \tag{63}$$

Note that the prefactor in Eq. (61) contains T_0 instead of T. The eigenvalues $\{\Lambda_k^n\}_{\rm min,sdl}$ are defined by Eqs. (51) and (54).

The remaining integral over B_1^0 in Eq. (61) can be expressed in terms of Fresnel integral,

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x dt\,\exp(-t^2) \tag{64}$$

and one obtains $\tau_{\rm esc}^{-1}$ in the final form

$$\tau_{\rm esc}^{-1}=\frac{1}{4}\sqrt{\frac{\pi}{B}}T_0\exp(-\Delta A_{\min}^{\rm sdl})Y_1Y_2\{\sqrt{\Lambda_0^0}\}_{\min}\cdot$$
$$\left\{\frac{1}{\sqrt{|\Lambda_0^0|}}\left[1-\Phi\left(\frac{\Lambda_1^0}{2\sqrt{B}}\right)\right]\exp\frac{(\Lambda_1^0)^2}{4B}\right\}_{\rm sdl}, \tag{65}$$

with

$$B=-\gamma_2+\gamma_1^2\left(\frac{2}{|\Lambda_0^0|}-\frac{1}{\Lambda_2^0}\right)_{\rm sdl}. \tag{66}$$

Using the explicit Eq. (54) for eigenfunctions Λ_n^0, one can present Eq. (63) in terms of Euler gamma function $\Gamma(x)$,

$$Y_2=\frac{4\hbar j_c\zeta}{e}\frac{\{\Gamma[2-n_1(H)]\Gamma[2-n_2(H)]\}_{\rm sdl}}{\{\Gamma[1-n_1(H)]\Gamma[1-n_2(H)]\}_{\min}},$$

while the values $[n_{1,2}]_{\rm saddle}$ and $[n_{1,2}]_{\min}$ are the roots of equation

$$\zeta(T)[n^2+Q_{\rm sh}(T)n]+\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0$$
$$+\left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0=0, \tag{67}$$

written for $\varphi_0=\{\varphi_0^{\rm saddle},\varphi_0^{\min}\}$ accordingly. From Eq. (67) one can see that the finiteness of L leads to strong variation of Y_2 and, consequently $\tau_{\rm esc}^{-1}$, as the function of magnetic field even for small junction with $L\ll\lambda_J$.

All quantities ($\Delta A_{\min}^{\rm sdl}$, T_0, and $\tau_{\rm esc}^{-1}$) are oscillatory functions versus magnetic field [see Eqs. (17), (30), (34), (44), (55), and (62)]. As the example let us consider the behavior of $T_0(H)$. In the first approximation by parameter $(L/\lambda_J)^2$, one can obtain from Eqs. (41)–(55) the following simple expression for the crossover temperature $T_0(H)$:

$$T_0^2+\varepsilon T_0-\frac{ej_cSR_{\rm sh}}{\pi}\varepsilon\sqrt{\delta_1^2(H)-\left(\frac{I}{j_cS}\right)^2-2\frac{\delta_2(H)}{\delta_1^2(H)}\left[\delta_1^2(H)-\left(\frac{I}{j_cS}\right)^2\right]}=0,$$

with $\varepsilon=Q_{\rm sh}T=\hbar/(2\pi R_{\rm sh}C^*)$ and δ_1 and δ_2 defined by Eq. (38). Its physical solution

$$T_0(H)=\frac{\varepsilon}{2}\left\{\left\{1+\frac{4ej_cSR_{\rm sh}}{\pi\varepsilon}\left\{\sqrt{\delta_1^2(H)-\left(\frac{I}{j_cS}\right)^2}-2\frac{\delta_2(H)}{\delta_1^2(H)}\left[\delta_1^2(H)-\left(\frac{I}{j_cS}\right)^2\right]\right\}\right\}^{1/2}-1\right\} \tag{68}$$

evidently oscillates versus the magnetic field as it is sketched in Fig. 3. Equation (68) can be used until the second term in square parenthesis is smaller than the first one. Close to the special points ($L=\pi nL_H$) this condition can not be valid

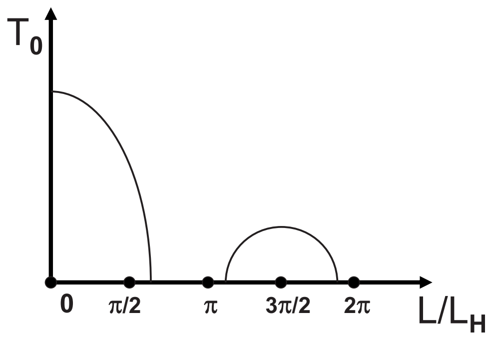

FIG. 3. Schematic dependence of crossover temperature on magnetic field.

more and in such a case Eqs. (30) and (67) should be solved exactly.

In contrast to Y_2, the prefactor Y_1 contains only terms with Λ_0^k $(k\neq 0)$ and in result depends on magnetic field weakly. Nevertheless this dependence turns out to be singular due to the logarithmic divergence of the product in Eq. (62). Details are presented in Appendix B.

Equation (65) enables us to estimate the width of the crossover region between Arrenius and macroscopic quantum tunneling (MQT) regimes. It can be found from the condition that the argument of the Fresnel function is of the order of one,

$$\{\Lambda_1^0\}_{\mathrm{sdl}} \sim 2\sqrt{B}. \tag{69}$$

VII. FINAL REMARKS

Even a weak external magnetic field can strongly change the value of the critical current of the JJ of a small size. Note that some special points of external magnetic field appear in the problem under consideration. In the vicinity of these points the perturbation theory fails and the escape time can be calculated only by means of the exact solution of the equations on T_0 and φ_0. It is worth mentioning that our general equations enable us to consider even such points.

In the vicinity of the crossover temperature T_0 from Arrenius law to the quantum tunneling, one can observe the strong effect of the finiteness of the junction length L even in the pre-exponential factor.

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APPENDIX A

The eigenvalues $\Lambda_{\pm 1}^0$ at the saddle-point trajectory tend to zero as $T\to T_0$. In result, when one substitutes the function $\varphi(\mathbf{r},\tau)$ in the form (20) to the action (45) and then expands the expression for action in Taylor series, he has to keep "dangerous" terms, containing $B_{\pm 1}^0$, up to the fourth order. Moreover one also has to keep products of the type $[(B_1^0)^2 B_{-2}^k,\ (B_{-1}^0)^2 B_2^k]$. All other terms (containing B_n^k with $k\neq 0$ and $n\neq 0,\pm 1$) are enough to take in the second-order approximation. In this way one writes the value of the effective action (45) valid near both extremal trajectories in the form (see Ref. 14)

$$A[\varphi(x,\tau)] = A[\varphi_{\mathrm{extr}}(x)] + \frac{1}{2}\sum_{n,k}\Lambda_n^k|B_n^k|^2 - \frac{2\hbar S j_c}{e}\sqrt{T}\int_{-L/2}^{L/2}\frac{dx}{L}\sin[2\varphi_{\mathrm{extr}}(x)]\left[2\frac{B_1^0B_{-1}^0\chi_1^0\chi_{-1}^0}{\|\chi_1^0\|\|\chi_{-1}^0\|}\sum_k\frac{B_0^k\chi_0^k}{\|\chi_0^k\|} + \frac{(B_1^0)^2(\chi_1^0)^2}{\|\chi_1^0\|^2}\sum_k\frac{B_{-2}^k\chi_{-2}^k}{\|\chi_{-2}^k\|}\right.$$
$$\left. + \frac{(B_{-1}^0)^2(\chi_{-1}^0)^2}{\|\chi_{-1}^0\|^2}\sum_k\frac{B_2^k\chi_2^k}{\|\chi_2^k\|}\right] - \frac{2\hbar S j_c}{e}T\int_{-L/2}^{L/2}\frac{dx}{L}\cos[2\varphi_{\mathrm{extr}}(x,\tau)]\frac{(B_1^0)^2(B_{-1}^0)^2}{\|\chi_1^0\|^2\|\chi_{-1}^0\|^2}(\chi_1^0)^2(\chi_{-1}^0)^2, \tag{A1}$$

where $\|..\|$ is the norm of the function. Let us carry out the integrals entering in the Eq. (A1) in the explicit form

$$I_1 = \int_{-L/2}^{L/2}\frac{dx}{L}\sin[2\varphi_{\mathrm{extr}}(x)]\frac{\chi_1^0\chi_{-1}^0\chi_0^0}{\|\chi_1^0\|\|\chi_{-1}^0\|\|\chi_0^0\|}$$
$$= \frac{1}{S^{3/2}}\left[\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\sin 2\varphi_0 + 2\left(\frac{L_H}{2\lambda_J}\right)^2\kappa\left(\frac{L_H}{L}\right)\sin 4\varphi_0\right] \tag{A2}$$

and

$$I_2 = \int_{-L/2}^{L/2}\frac{dx}{L}\cos(2\varphi_{\mathrm{extr}}(x))\frac{(\chi_1^0)^2(\chi_{-1}^0)^2}{\|\chi_1^0\|^2\|\chi_{-1}^0\|^2}$$
$$= \frac{1}{S^2}\left\{\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)\cos 2\varphi_0 + \frac{1}{2}\left(\frac{L_H}{2\lambda_J}\right)^2\left[5\kappa\left(\frac{L_H}{L}\right)\cos 4\varphi_0 + 3\mu\left(\frac{L_H}{L}\right)\right]\right\}, \tag{A3}$$

with the function μ defined by Eq. (35). Using these expressions in Eq. (A1) one can obtain the final expression (56) for the effective action valid for trajectories close to the extremal ones. Integrals (A2) and (A3) appear in Eq. (56) by means of the functions (57) and (58)

$$\gamma_1 = \frac{2\hbar j_c S}{e}\sqrt{T}I_1, \quad \gamma_2 = \frac{2\hbar j_c ST}{e}I_2. \tag{A4}$$

APPENDIX B

One can see that expression for the pre-exponential factor Y_1 (62) is divergent at $\{n,k\}\to\infty$ [this follows from Eq. (53)]. This divergency has the logarithmic character and should be cut off at $n\sim T_c/T_0$ and $k\sim\lambda_J/\ell_{\rm eff}$. As a result with the logarithmic accuracy, one can find,

$$Y_1 = \left[\prod_{k=1}\frac{1}{\sqrt{\Lambda_0^k}}\left(\prod_{n=1}^{\infty}\frac{1}{2\Lambda_n^k}\right)\right]_{\rm sdl}\left[\left(\prod_{k=1}^{\infty}\frac{1}{\sqrt{\Lambda_0^k}}\prod_{n=1}^{\infty}\frac{1}{2\Lambda_n^k}\right)\right]_{\rm min}^{-1}$$
$$= \exp\left\{-\frac{1}{2}\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)[\cos(2\varphi_0^{\rm sdl})-\cos(2\varphi_0^{\rm min})]\sum_{k=1}^{\infty}\frac{1}{1-\left(\frac{L_H}{L}\pi k\right)^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\frac{\pi k\lambda_J}{L}\right)^2+\zeta(n^2+Q_{\rm sn}|n|)}\right\}$$
$$\exp\left\{-\frac{1}{2}\frac{L_H}{L}\sin\left(\frac{L}{L_H}\right)[\cos(2\varphi_0^{\rm sdl})-\cos(2\varphi_0^{\rm min})]\right\}\sum_{k=1}^{\lambda_J/\ell_{\rm eff}}\sum_{n=-T_c/T_0}^{T_c/T_0}\left[\left(\frac{\pi k\lambda_J}{L}\right)^2+\zeta(n^2+Q_{\rm sn}|n|)\right]^{-1}. \tag{B1}$$

The double sum in the first multiplier in the right-hand side of this expression converges at infinity.

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